Torsion Zero-Cycles on Abelian Surfaces
Obtained by Weil Restriction of
Modular Elliptic Q-Curves over Quadratic Fields (I)

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1. Introduction.

Chow groups of varieties are a natural generalization of Weil divisor class groups to higher codimensional case. Algebraic cycles and rational equivalence correspond Weil divisors and linear equivalence respectively. The latter is codimension-one case. As is well known, Weil divisor class groups of non-singular varieties over number fields are finitely generated abelian groups by the theorems of Mordell-Weil and Néron-Severi, but Chow groups of higher codimension are not known to be finitely generated.

In this paper we will consider the result of Mildenhall [11], where he studied the second Chow group of self-product of a modular elliptic curve $E_Q$ over rational number field $Q$. It follows from his result that $n$-torsion subgroup of the second Chow group is finite for any fixed integer $n \neq 1$. In his proof a modular parametrization of an elliptic curve over $Q$ by a modular curve $X_0(N)$ is crucial. The existence of such a parametrization, which is a version of the Taniyama-Shimura conjecture, is proved by Taylor and Wiles for semi-stable case in 1995.

Expecting an analogous result we will study a case of an elliptic $Q$-curve $E$ over a quadratic field $K$. Here an elliptic $Q$-curve is an elliptic curve which is isogenous over $Q$ to any its galois conjugate $E^\sigma$, $\sigma \in \text{Gal}(\overline{Q}/Q)$ [7]. Let $A$ be the Weil restriction $\text{Res}_{K/Q}(E)$ of a $Q$-curve $E$. Then $A$ is an abelian surface defined over $Q$, which is isomorphic over $K$ to $E \times E^\sigma$, $(\sigma) = \text{Gal}(K/Q)$ (i.e. $A \otimes K \simeq E \times E^\sigma$) and there is an isomorphism $A(Q) \simeq E(K) \simeq (E \times E^\sigma)^{(\sigma)}$ (see [23]). Such an abelian variety $A$ is expected to appear as a $Q$-simple factor of the jacobian variety $J_1(N)$ of a modular curve $X_1(N)$ up to isogeny over $Q$. This phenomenon is an extension of the Taniyama-Shimura conjecture in some sense (Elliptic curves over $Q$ are trivially $Q$-curves). If this occurs, $E$ has a modular parametrization by $X_1(N)$. Recently Hasegawa-Hashimoto-Momose [9] gave a criterion for modularity of elliptic $Q$-curves. Moreover Hasegawa [8] made families of elliptic $Q$-curves with one parameter in their coefficients over various quadratic fields. Especially he made one parameter families of $Q$-curves which are isogenous to its galois conjugate over the field of definition of the curves. Using modular parametrizations for these $Q$-curves, we obtain the following theorem:

Theorem A. Let $E$ be a modular elliptic $Q$-curve over a quadratic field $K$. Assume that $E$ is isogenous over $K$ to its galois conjugate $E^\sigma$ and that $E$ satisfies a certain condition $(*)$ stated later. Let $A := \text{Res}_{K/Q}(E)$ be its Weil restriction. Then for
any fixed integer \( n \neq 1 \) \( n \)-torsion subgroup \( n \text{CH}^2(A) \) of the second Chow group \( \text{CH}^2(A) \) is finite.

As for torsion part of Chow groups of varieties over number fields, the most general result known so far is due to Colliot-Thélène and Raskind [2] and Salberger [18]. They showed that for any projective smooth variety \( X \) over a number field the torsion subgroup \( \text{CH}^2(X) \) of the second Chow group \( \text{CH}^2(X) \) is finite, under an assumption \( H^2_{\text{zar}}(X, \mathcal{O}_X) = 0 \). The assumption was used essentially to lift line bundles in their proof (see [2]), but examples for the case \( H^2_{\text{zar}}(X, \mathcal{O}_X) \neq 0 \) were made in these years. That is, if \( X \) is the self-product of a modular elliptic curve \( E_\mathbb{Q} \) or the Fermat quartic surface, the \( p \)-primary torsion subgroup \( \text{CH}^2(X) \) is finite for almost all prime numbers \( p \). These are works of Langer and Saito [10] and Otsubo [15]. Our result theorem A corresponds to the “théorème A” of [2] which says the finiteness of \( n \)-torsion. As is proved in [2], our result should be extended to the result of finiteness of \( p \)-primary torsion. This will be treated in the sequel paper (II).

As for free part of Chow groups, the Bass conjecture tells that \( K_0(\mathcal{X}) \) is finitely generated for \( \mathcal{X} \) a regular scheme of finite type over \( \mathbb{Z} \). This means, combined with Grothendieck isomorphism \( K_0(\mathcal{X}) \otimes \mathbb{Q} \simeq (\bigoplus_i \text{CH}^i(\mathcal{X})) \otimes \mathbb{Q}, \) that the Chow group \( \text{CH}^i(\mathcal{X}) \) has a finite rank. Moreover the rank of the Chow group \( \text{CH}^i(X) \) for \( X \) a variety over a number fields is expected to coincide to the order at a certain integral point of the \( L \)-function related to it (see [17]). This is an extension of the Birch and Swinnerton-Dyer conjecture for (\( H^1_{\text{et}} \)) of abelian varieties over number fields.

This paper proceeds as follows. In section 1, we consider an exact sequence called Sherman’s localization sequence, which is a basic tool through the paper:

\[
H^1(A, \mathcal{K}_2) \rightarrow H^1(A, \mathcal{K}_2) \xrightarrow{\partial} \bigoplus_{p \in \text{Spec}\mathbb{Z}[1/N]} \text{Pic}(A_p) \quad \rightarrow \quad \text{CH}^2(A) \xrightarrow{\beta} \text{CH}^2(A) \rightarrow 0.
\]

Here cohomologies are taken over Zariski topology, \( A \) is the Néron model of \( A \) over \( \text{Spec}\mathbb{Z}[1/N] \) and \( N \) is the square root of the conductor of \( A/\mathbb{Q} \). Here the conductor is known to be a square number.

We want to show that \( n \text{CH}^2(A) \) is finite if the kernel of the map \( \beta \) is torsion, so the problem is reduced to the investigation of the cokernel of the map \( \partial \). Then we know the structure of the Néron-severi group \( \text{NS}(A_p) \), where \( A_p \) is the reduction of \( A \) at a rational prime \( p \). This is difficult, and for this reason we consider \( \text{NS}(E_p \times E_p)^{\text{Gal}(\bar{F}_p/F_p)} \) rather than \( \text{NS}(A_p) \), where \( p \) is a fixed prime ideal in \( K \) above \( p \). Then we apply the method of Mildenhall [11] and Flach [5] in section 2. That is, we construct an element in \( H^1(E \times E^\times, \mathcal{K}_2) \) which maps to a certain multiple of the gragh of the Frobenius endomorphism in \( E_p \times E_p^\times \) under the
map \( \partial \). This element is made at first in \( H^1(X_0(N) \times X_0(N), K_2) \) by the Eichler-Shimura congruence relation and carried to \( H^1(E \times E^\sigma, K_2) \) through the modular parametrization. Then we finish the proof that \( \text{Coker}(\partial) \) is torsion.

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2. The first reduction to investigate the \( n \)-torsion subgroup.

Let \( E \) be an elliptic \( \mathbb{Q} \)-curve over a quadratic field \( K \). Let \( A := \text{Res}_{K/\mathbb{Q}}(E) \) be its Weil restriction. \( A \) is an abelian surface over \( \mathbb{Q} \) which splits to \( E \times E^\sigma \) over \( K \) by definition [23], i.e.

\[
A \otimes K \simeq E \times E^\sigma.
\]

The conductor of \( A/\mathbb{Q} \) is a square integer \( N^2 \), more precisely

\[
\text{cond}(A) = \text{cond}(\text{Res}_{K/\mathbb{Q}}(E)) = N_{K/\mathbb{Q}}(\text{cond}(E)) \cdot \text{disc}(K/\mathbb{Q})^2 = N^2
\]

for some integer \( N \) (see [12]). In [9], it is shown that \( A \) appears as a \( \mathbb{Q} \)-simple factor up to isogeny over \( \mathbb{Q} \) of the jacobian variety \( J_1(N) \) of a modular curve \( X_1(N) \) under certain conditions. If the surjective map \( J_1(N) \twoheadrightarrow A \) exists, there is a natural map:

\[
X_1(N) \otimes K \to J_1(N) \otimes K \to E \times E^\sigma,
\]

which induces the modular parametrization

\[
X_1(N) \otimes K \to E.
\]

We assume \( E \) to be modular in the sense that \( E \) has this modular parametrization by \( X_1(N) \). Moreover we will need that \( E \) is isogenous to \( E^\sigma, \langle \sigma \rangle = \text{Gal}(K/\mathbb{Q}) \), over the field of definition \( K \) of \( E \):

\[
E \bigotimes_{K/\mathbb{Q}} E^\sigma.
\]

Such elliptic \( \mathbb{Q} \)-curves over quadratic fields \( K \) are given in [8] as one parameter families. We show in the following the finiteness of the \( n \)-torsion subgroup \( n \text{CH}^2(A) \) of the second Chow group of the abelian surface \( A \) obtained from such \( E \). The conditions that \( E \) is modular and that \( E \) is isogenous over \( K \) to \( E^\sigma \) are used in the next section.

In order to apply \( K \)-theory, let \( A \) be the Néron model of \( A \) over the ring \( \mathbb{Z}[1/N] \), which is proper smooth regular model over \( \mathbb{Z}[1/N] \). Let \( A_p \) denote a closed fiber over \( p \); the reduction modulo \( p \). Gersten’s conjecture holds for \( A \) and Sherman’s localization sequence [2, Lem.3.2] is written down as follows:

\[
\begin{align*}
H^1(A, K_2) & \to H^1(A, K_2) \xrightarrow{\partial} \bigoplus_{p \in \text{Spec} \mathbb{Z}[1/N]} H^1(A_p, K_1) \to H^2(A, K_2) \xrightarrow{\beta} H^2(A, K_2) \to 0.
\end{align*}
\]
Applied the Bloch’s formula [16], the sequence above turns into the following:

\[
H^1(A, \mathcal{K}_2) \to H^1(A, \mathcal{K}_2) \otimes p \in \text{Spec} \mathbb{Z}[1/N] \bigoplus \text{Pic}(A_p) \\
\to \text{CH}^2(A) \overset{\beta}{\to} \text{CH}^2(A) \to 0.
\]

**Proposition 2.1.**

In the above sequence, \( \sigma \text{CH}^2(X) \) is finite if \( \text{Ker}(\beta)(=\text{Coker}(\partial)) \) is torsion.

**Proof.** Let \( n \neq 1 \) be an integer and put \( A' := A \otimes_{\mathbb{Z}[1/N]} \mathbb{Z}[1/nN] \) and \( \sum(A) := \text{Ker}(\beta) \). Consider the following commutative diagram (which defines \( \sum(A') \)):

\[
\begin{array}{cccccc}
0 & \longrightarrow & \sum(A) & \longrightarrow & \text{CH}^2(A) & \overset{\beta}{\longrightarrow} & \text{CH}^2(A') & \longrightarrow & 0 \\
\downarrow & & \downarrow n_a & & \downarrow n_b & & \downarrow n_c & & \downarrow \\
0 & \longrightarrow & \sum(A') & \longrightarrow & \text{CH}^2(A') & \longrightarrow & \text{CH}^2(A) & \longrightarrow & 0.
\end{array}
\]

Here \( (n)_a, (n)_b \) and \( (n)_c \) are the multiplication-by-\( n \) map, and the middle vertical map is induced by deleting the fibers above \( n \). If \( \sum(A) \) is torsion, the cokernel of \( (n)_a \) is isomorphic to \( n \)-torsion subgroup of \( \sum(A_0) \). On the other hand, the theorem of Gillet says as follows:

**Theorem 2.2 (Gillet)**[2, Thm.1.1]. Let \( L \) be a number field, \( \mathcal{O} \) be its integer ring. Let \( X \) be a smooth \( \mathcal{O}[1/f] \)-scheme, \( f \in \mathcal{O}, f \neq 0 \) and \( n \) be an integer invertible in \( \mathcal{O}[1/f] \). Then \( \sigma \text{CH}^2(X) \) is finite.

See [2, Thm.1.1] for the proof. Then \( \sigma \text{CH}^2(A) \) is finite by this theorem, so \( \sigma \sum(A') \) is finite. Using the snake lemma, \( \text{Ker}(\partial) = \sigma \text{CH}^2(A) \) is finite. \( \square \)

By the proposition above, we can replace the problem to prove the finiteness of \( n \)-torsion with the research of the map \( \partial \). The map \( \partial \) is precisely the following ([2, Sect.3]):

\[
\partial : H^1(A, \mathcal{K}_2) \to \bigoplus_{p \in \text{Spec} \mathbb{Z}[1/N]} \text{Pic}(A_p),
\]

\[
[\sum_D(D, f_D)] \mapsto \sum_{p|N} \left[ \sum_D \text{div}(f_{D_p}) \right],
\]

where \( D \) is a divisor on \( A \), \( f_D \) is a rational function on \( D \) satifying \( \sum_D \text{div}(f_D) = 0 \). \( f_{D_p} \) is the extended function of \( f_D \) to on \( D \), the closure of \( D \) in \( A \). \( f_{D_p} \) is the restricted function of \( f_{D_p} \) to on \( D_p \), the fiber above \( p \).

There is a decomposition of the Picard group:

\[
\text{Pic}(A_p) \simeq \text{Pic}^0(A_p) \oplus \text{NS}(A_p).
\]

Here \( \text{Pic}^0(A_p) = \widehat{A_p}(\mathbb{F}_p) \), the \( \mathbb{F}_p \)-rational points of the dual abelian variety of \( A_p \) [13, Sect.9]. In particular \( \text{Pic}^0(A_p) \) is torsion. So we concentrate on \( \text{NS}(A_p) \) hereafter.
The Néron-Severi group NS(A_p) is the group of divisors modulo algebraic equivalence. To investigate this explicitly, extend the base of A to O_K[1/N]. It splits to the product of the Néron model E of E over O_K[1/N] and its galois conjugate (O_K is the integer ring of K):

\[ A \times_{\text{Spec} \mathbb{Z}[1/N]} \text{Spec} O_K[1/N] = E \times_{\text{Spec} O_K[1/N]} E^\sigma. \]

Take a prime p over p and fix it. The generic fiber is A \otimes K and a closed fiber over p is the reduction modulo p:

\[ A \times_{\text{Spec} \mathbb{Z}[1/N]} \text{Spec} O_K[1/N] \times_{\text{Spec} O_K[1/N]} \text{Spec} K \]
\[ = A \otimes K = E \times E^\sigma, \]
\[ A \times_{\text{Spec} \mathbb{Z}[1/N]} \text{Spec} O_K[1/N] \times_{\text{Spec} O_K[1/N]} \text{Spec} \mathbb{F}_p \]
\[ = (A \otimes K)_p = E_p \times E_p^\sigma. \]

Denote the galois group Gal(\mathbb{F}_p/\mathbb{F}_p) by \( \langle \sigma_p \rangle \). If \( \sigma \) is a rational point, i.e. zero-elements of \( \mathbb{Z}/2\mathbb{Z} \) if \( p \) is inert to \( p = pO_K \). Taking the galois cohomology of the exact sequence below

\[ 0 \rightarrow \text{Pic}^0(E_p \times E_p^\sigma) \rightarrow \text{Pic}(E_p \times E_p^\sigma) \rightarrow \text{NS}(E_p \times E_p^\sigma) \rightarrow 0 \]
yields the upper exact sequence in the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Pic}^0(E_p \times E_p^\sigma) & \longrightarrow & \text{Pic}(E_p \times E_p^\sigma) & \longrightarrow & \text{NS}(E_p \times E_p^\sigma) \\
\uparrow & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong \\
0 & \longrightarrow & \text{Pic}^0(A_p) & \longrightarrow & \text{Pic}(A_p) & \longrightarrow & \text{NS}(A_p) \end{array}
\]

The vertical isomorphisms are valid by the following equations (here we use the fact that A has at least one \( \mathbb{Q} \)-rational point, i.e. zero-element):

\[
\text{Pic}^0((A \otimes K)_p)^{\langle \sigma_p \rangle} = (A \otimes K)_p(\mathbb{F}_p)^{\langle \sigma_p \rangle} = \tilde{A_p}(\mathbb{F}_p) = \text{Pic}^0(A_p),
\]
\[
\text{Pic}((A \otimes K)_p)^{\langle \sigma_p \rangle} = \text{Pic}(A_p),
\]

Moreover the injective map \( \text{NS}(A_p) \rightarrow \text{NS}(E_p \times E_p^\sigma)^{\langle \sigma_p \rangle} \) is actually an isomorphism

\[ \text{NS}(A_p) \cong \text{NS}(E_p \times E_p^\sigma)^{\langle \sigma_p \rangle} \]
because \( H^1((\langle \sigma_p \rangle, E_p \times E_p^\sigma) = 0 \) (the obstruction \( H^1(\text{Gal}(\overline{L}/L), \text{Pic}^0(X/L \otimes \overline{L}) \) is always trivial if X is a variety over any finite field L [13][14]).

Moreover \( \text{NS}(E_p \times E_p^\sigma) \) has a decomposition:

\[ \text{NS}(E_p \times E_p^\sigma) \cong \text{NS}(E_p) \oplus \text{NS}(E_p^\sigma) \oplus \text{DC}_{\mathbb{F}_p}(E_p, E_p^\sigma)_{\text{alg}}. \]

Here \( \text{NS}(E_p) \cong \mathbb{Z} \), the isomorphism being given by \( D \mapsto \deg(D) \), here D is a divisor. Hence \( \text{NS}(E_p) \) is generated by an algebraic equivalence class of divisors of degree 1. Especilly we can take the class of origin \( O_p \) of \( E_p \) as its generator.

\[ \text{DC}_{\mathbb{F}_p}(E_p, E_p^\sigma)_{\text{alg}} \] denotes a free abelian group on the set of divisorial correspondences defined over \( \mathbb{F}_p \) between two pointed schemes \( E_p \) and \( E_p^\sigma \) modulo algebraic equivalence, “pointed” means that \( E_p \) and \( E_p^\sigma \) have distinguished K-rational points, i.e. zero-elements \( O_p \) and \( O_p^\sigma \). (In general, a divisorial correspondence between two pointed schemes \( (X_1, P_1) \) and \( (X_2, P_2) \) over the same base field
There is a one-to-one correspondence between isomorphism classes $DC_{F_p}(E_p, E_p^\sigma)$ and any $\gamma$. There is a one-to-one correspondence between $DC_{F_p}(E_p, E_p^\sigma)$ and $\gamma$. Graph morphisms $Hom_{F_p}(E_p, E_p^\sigma)$ are defined over the same base field $F_p$, in particular, the graph $G_{\phi_p}$ of $\phi_p \in Hom_{F_p}(E_p, E_p^\sigma)$ is rationally equivalent to the graph $G_{\tilde{\phi}_p}$ of $\tilde{\phi}_p \in Hom_{F_p}(E_p, E_p^\sigma)$ (see [13, Cor.6.3]) ($\tilde{\phi}$ denotes the dual isogeny of $\phi$). This is also true in the algebraic equivalence situation among $DC_{F_p}(E_p, E_p^\sigma)$. That is, there is a one-to-one correspondence between $DC_{F_p}(E_p, E_p^\sigma)$ and $Hom_{F_p}(E_p, E_p^\sigma)$.

The argument above is summarized as follows. There is a decomposition

$$Pic(A_p) \simeq Pic^0(A_p) \oplus (NS(E_p) \oplus NS(E_p^\sigma) \oplus DC_{F_p}(E_p, E_p^\sigma))^{\sigma_p},$$

and any $[C_p]_{rat} \in Pic(A_p)$ is written as the following form:

$$[C_p]_{rat} = [C_p^0]_{rat} + [V_p]_{alg} + [H_p]_{alg} + [G_p]_{alg},$$

where

$$[C_p^0]_{rat} = (a priori torsion cycle class) \in Pic^0(A_p),$$

$$[V_p]_{alg} = (vertical cycle class) = [l \cdot O_p]_{alg} \times [E_p^\sigma]_{alg}, l \in \mathbb{Z},$$

$$[H_p]_{alg} = (horizontal cycle class) = [E_p]_{alg} \times [m \cdot O_p^\sigma]_{alg}, m \in \mathbb{Z},$$

$$[G_p]_{alg} = (graph cycle class) = [n \cdot (\text{the graph of } \phi_p)]_{alg} = [n \cdot (\text{the graph of } \tilde{\phi}_p)]_{alg},$$

$$\phi_p \in Hom_{F_p}(E_p, E_p^\sigma), \quad \tilde{\phi}_p \in Hom_{F_p}(E_p^\sigma, E_p), n \in \mathbb{Z},$$

such that the following equality holds:

$$([V_p]_{alg} + [H_p]_{alg} + [G_p]_{alg})^{\sigma_p} = [V_p]_{alg} + [H_p]_{alg} + [G_p]_{alg}.$$

Let’s consider the galois fixed part in the next section.

3. The Néron-Severi group of the Weil restriction.

Firstly we consider “global cycles”. There is a natural map induced by the product of $K$-groups:

$$Pic(E \times E^\sigma) \otimes_{\mathbb{Z}} K^* \overset{val}{\rightarrow} H^1(E \times E^\sigma, \mathcal{K}_2).$$

Compose this with the map $\partial$, then we obtain a map:

$$Pic(E \times E^\sigma) \otimes_{\mathbb{Z}} K^* \overset{val}{\rightarrow} H^1(E \times E^\sigma, \mathcal{K}_2) \overset{\partial}{\rightarrow} \bigoplus_{p \in Spec O_K[1/N]} Pic(E_p \times E_p^\sigma).$$

This composite map $\partial \circ val = \{ \partial_p \circ val_p \}_p$ is interpreted as the reduction at $p$ of the cycle in $Pic(E \times E^\sigma)$ equipped with coefficients of valuations at $p$ [2, Lem.3.2].

A class of cycles contained in the image of this map $\partial \circ rv$ is called the global cycle class. Let $h$ be the class number of the field $K$, then if $[C_p]_{rat} \in Pic(E_p \times E_p^\sigma)$ is obtained by specialization of $[C_p]_{rat} \in Pic(E \times E^\sigma)$, $hC_p$ is global because $p^h$ is a principal ideal.
Secondly, analogous to local case, there is a decomposition of the Picard group
\[ \text{Pic}(E \times E) \simeq \text{Pic}^0(E \times E') \oplus \text{NS}(E) \oplus \text{NS}(E') \oplus DC_K(E, E')_{\text{alg}}. \]
Now recall the assumption that \( \text{Hom}_K(E, E') \neq \{0\} \), that is, there are really the
local graph cycles. More precisely, they are write down as follows:
\[ [G]_{\text{alg}} = ([n \cdot \text{the graph of } \phi])_{\text{alg}} = [n \cdot \text{the graph of } \hat{\phi}]_{\text{alg}}, \]
\[ \phi \in \text{Hom}_K(E, E'), \hat{\phi} \in \text{Hom}_K(E', E), n \in \mathbb{Z}. \]
Comparing this with the local graph cycle, we see that there are cycles which are
not global in \( DC_{\mathbb{F}_p}(E_p, E'_p)_{\text{alg}} \) if the rank of homomorphism group \( \text{Hom}_K(E, E') \)
becomes larger by the reduction modulo \( p \), i.e. to show that \( \text{Coker}(\partial) \) is torsion,
we must kill the cycle which appear by the reduction (the problem of comparing
the global and local Picard number. This is the main argument in the rest of
this section). For the CM (complex multiplication)-elliptic curves over \( \mathbb{Q} \), there are
half primes which cause supersingular reduction \([19, V.\text{Exm.4.5]}\), and for non-CM-
elliptic curves over \( \mathbb{Q} \), Serre showed that the set of such “supersingular primes” has
density zero in the set of all primes, but Elkies proved that there are infinite number
of supersingular primes (see [4] for this matter).
On the other hand, there are morphisms below
\[ X_0(N) \otimes K \twoheadrightarrow X_1(N) \otimes K \xrightarrow{\psi} E. \]
\( \psi \) is the modular parametrization. For convention, Put \( X_0^2 := X_0(N) \times X_0(N), X_1^2 := X_1(N) \times X_1(N) \) and \( E^2 := E \times E \). The morphisms above induce the
following commutative diagram
\[
\begin{array}{c}
H^1(X_0^2, K_2) \xrightarrow{\partial} \bigoplus_{p|N \Pi} \text{Pic}((X_0^2)_{/p}) \\
\uparrow N_{K/\mathbb{Q}} \quad \uparrow N_{\mathbb{F}_p/\mathbb{F}_p} \\
H^1(X_0^2 \otimes K, K_2) \xrightarrow{\partial} \bigoplus_{p|N \Pi} \text{Pic}((X_0^2 \otimes K)_{/p}) \\
\uparrow \\
H^1(X_1^2 \otimes K, K_2) \xrightarrow{\partial} \bigoplus_{p|N \Pi} \text{Pic}((X_1^2 \otimes K)_{/p}) \\
\uparrow \psi \times \psi \\
H^1(E^2, K_2) \xrightarrow{\partial} \bigoplus_{p|N \Pi} \text{Pic}(E^2_{/p}) \\
\uparrow (\text{id} \times \phi) \\
H^1(E \times E', K_2) \xrightarrow{\partial} \bigoplus_{p|N \Pi} \text{Pic}(E_p \times E'_p) \\
\uparrow N_{K/\mathbb{Q}} \quad \uparrow N_{\mathbb{F}_p/\mathbb{F}_p} \\
H^1(A, K_2) \xrightarrow{\partial} \bigoplus_{p|N \Pi} \text{Pic}(A_p). \\
\end{array}
\]
Here \( N_{K/\mathbb{Q}} \) and \( N_{\mathbb{F}_p/\mathbb{F}_p} \) is the induced norm maps and \( \phi \) is the isogeny which is the base of \( \mathbb{Z} \subseteq \text{Hom}_K(E, E') \).
The situation in the galois fixed part is different according to whether $p$ splits into $p$ and $p^\sigma$ in $K$ or $p$ is inert to $p\mathcal{O}_K = p$.

(1) *Split case*: $p = pp^\sigma$.

Let $F_p = F_p$, so $E_p$ and $E_p^\sigma$ are defined over $F_p$ and the galois group $\langle \sigma_p \rangle$ is trivial. Observe the cycle classes in the following:

$$\begin{align*}
(Pic^0(E \times E^\sigma) \oplus NS(E) \oplus NS(E^\sigma) \oplus DC_K(E, E^\sigma)_{alg}) \otimes K \\
\bigoplus_{p = p \text{ or } p^\sigma}(Pic^0(E_p \times E_p^\sigma) \oplus NS(E_p) \oplus NS(E_p^\sigma) \oplus DC_{F_p}(E_p, E_p^\sigma)_{alg}) \\
\downarrow_{N_{F_p}/F_p}\downarrow_{\sigma_p}

\text{Pic}^0(A_p) \oplus NS(E_p) \oplus NS(E_p^\sigma) \oplus DC_{F_p}(E_p, E_p^\sigma)_{alg}.
\end{align*}$$

In the map above, $[G_p]_{rat}$ is a priori torsion, and we can see that $2h \cdot [V_p]_{alg}$ and $2h \cdot [H_p]_{alg}$ are global, so $[V_p]_{alg}$ and $[H_p]_{alg}$ are torsion in $\text{Coker}(\partial)$. 2-multiple occurs because of the norm argument ($[K : Q] = 2$).

Let's consider the graph cycle class $[G_p]_{alg}$. If $E$ is CM-elliptic curve, this is treated in [11] and [15] (which treat both cases that $p$ splits and $p$ is inert in $K$). So we discuss only non-CM case hereafter.

Fix the isogeny $\phi \in \text{Hom}_K(E, E^\sigma)$ which is the $Z$-basis of $\text{Hom}_K(E, E^\sigma) \simeq Z$. Let $[n]$ denote the multiplication-by-$n \in \text{End}_K(E)$, let $\phi_p$ and $[n]_p$ be the reduction-isogeny modulo $p$, and $\pi_p \in \text{End}_{F_p}(E_p)$ the $p$-power Frobenius associated $F_p \simeq F_p$.

Note that if a certain multiple of the graph $G_{[n]_p}$ of $[n]_p$ are global, the problem whether or not the graph cycle class $[G_{[n]_p}]_{alg}$ of the composite isogeny $f_p \circ [n]_p \in \text{Hom}_{F_p}(E_p, E_p^\sigma)$ are torsion in $\text{Coker}(\partial)$ is reduced to that of $[G_{[n]_p}]_{alg}$. But $2h \cdot G_{[n]_p}$ is global because which comes from $\text{End}_K(E) \simeq Z$. Therefore we only have to verify that the graph cycle classes are torsion for two or four isogenies which form a $Z$-basis of $\text{Hom}_{F_p}(E_p, E_p^\sigma)$. A remarkable fact in the split case is that the rank of $\text{Hom}_{F_p}(E_p, E_p^\sigma)$ is always two because even if the quaternions exists in $\text{End}_{F_p}(E_p)$, they are not defined over $F_p = F_p$ so are not in $\text{End}_{F_p}(E_p)$. This is explained as follows. In $\text{End}_{F_p}(E_p) \otimes Q$, we can see $Q[\pi_p]$ has dimension two over $Q$ because $\pi_p$ has the degree $p$ and this is not a square-integer. On the other hand, the centralizer of $Q[\pi_p]$ in $\text{End}_{F_p}(E_p) \otimes Q$ is $\text{End}_{F_p}(E_p) \otimes Q$, i.e. defined over the field of definition $F_p$. The theorem of Brauer says that the centralizer has dimension two, so $\text{End}_{F_p}(E_p)$ coincides with $Q[\pi_p]$ (more precisely, see [22]). Hence we consider the graph of $\phi_p$ and $\phi_p \circ \pi_p$.

Firstly for the graph $G_{\phi_p}$, $2h \cdot G_{\phi_p}$ is global. Next consider the graph $G_{\phi_p \circ \pi_p}$. We can kill some multiple of $G_{\phi_p \circ \pi_p}$ by the elements in $H^1(A, \mathcal{K}_2)$ which is constructed at first in $H^1(X_{\beta}^2, \mathcal{K}_2)$ by Mildenhall and Flach:

**Theorem 3.1 (Mildenhall [11] and Flach [5]).**

*For each prime $p \nmid N$, there is an element $x_p \in H^1(X_{\beta}^2, \mathcal{K}_2)$ which maps under the*
boundary map $\partial_p$ to $k$-multiple of the graph of $p$-power Frobenius

$$\partial_p : H^1(X_0^2, \mathcal{K}_2) \to \bigoplus_{p|N} \text{Pic}((X_0^2)_p)$$

$$x_p \mapsto k[G_{\phi_p}]_{\text{alg}}.$$ 

$k$ is the weight of a certain modular function on $X_0(pN)$. $X_0(pN)$ is the proper model of $X_0(pN)$ over $\mathbb{Z}[1/N]$ which is smooth outside $p$.

For the outline of the proof of this theorem, see appendix of this paper.

Let $a_p$ be the corresponding element in $H^1(A, \mathcal{K}_2)$ obtained by the pull-back and the push-forward in the commutative diagram before. Then $\partial_p(a_p)$ maps to $2k$-multiple of $[G_{\phi_p}]_{\text{alg}}$:

$$H^1(A, \mathcal{K}_2) \to \bigoplus_{p|N} \text{Pic}(A_p)$$

$$a_p \mapsto 2k[G_{\phi_p}]_{\text{alg}}.$$ 

This completes the claim:

$$\text{Coker}(\partial_p) := \text{Coker}(\partial_p : H^1(A, \mathcal{K}_2) \to \text{Pic}(A_p))$$

is torsion for any $p$ which splits in $K$.


$F_p = F_{p^2}$, so $E_p$ and $E_p^\sigma$ are defined over $F_{p^2}$ and the galois group $\langle \sigma_p \rangle \simeq \mathbb{Z}/2\mathbb{Z} \simeq \langle \sigma \rangle$. Observe the cycle classes in the following:

$$\begin{align*}
(\text{Pic}^0(E \times E^\sigma) \oplus \text{NS}(E) \oplus \text{NS}(E^\sigma) \oplus \text{DC}_K(E, E^\sigma)_{\text{alg}}) \otimes K & \xrightarrow{\partial_p \circ \text{val}_p} \\
\text{Pic}^0(E_p \times E_p^\sigma) \oplus \text{NS}(E_p) \oplus \text{NS}(E_p^\sigma) \oplus \text{DC}_{F_p}(E_p, E_p^\sigma)_{\text{alg}} & \xrightarrow{N_{E_p/F_p}} \\
\text{Pic}^0(A_p) \oplus (\text{NS}(E_p) \oplus \text{NS}(E_p^\sigma) \oplus \text{DC}_{F_p}(E_p, E_p^\sigma)_{\text{alg}})^{\langle \sigma_p \rangle}. &
\end{align*}$$

The galois fixed part accurs in this case. It is decomposed to the (vertical and horizontal part) and (graph part):

$$(\text{NS}(E_p) \oplus \text{NS}(E_p^\sigma) \oplus \text{DC}_{F_p}(E_p, E_p^\sigma)_{\text{alg}})^{\langle \sigma_p \rangle}$$

$$\simeq (\text{NS}(E_p) \oplus \text{NS}(E_p^\sigma))^{\langle \sigma_p \rangle} \oplus \text{DC}_{F_p}(E_p, E_p^\sigma)^{\langle \sigma_p \rangle}.$$ 

We can see that $2h$-multiple of the cycle classes in $(\text{NS}(E_p) \oplus \text{NS}(E_p^\sigma))^{\langle \sigma_p \rangle}$ is global.

A remarkable fact in the inert case is that we may have to treat the graphs of the quaternion $\in \text{Hom}_{F_p}(E_p, E_p^\sigma)$ because if non-CM elliptic curve $E$ over $K$ has a supersingular reduction at $p$ (which is quite rare stated before), the isogenies corresponding to the quaternions may be defined over $F_p = F_{p^2}$. Hence we must watch carefully the galois fixed part of the graphs in $\text{DC}_{F_p}(E_p, E_p^\sigma)_{\text{alg}}$.

The galois action on the Weil restriction is given as follows. For a point $(P_1, P_2)$ in $E \times E^\sigma$ $\sigma$ acts by $(P_1, P_2)^\sigma = (P_2^\sigma, P_1^\sigma)$ ([23]). The action of $\langle \sigma_p \rangle$ on $E_p \times E_p^\sigma$ is the same but with the compatibility with reductions. If $f$ is an isogeny in $\text{Hom}_K(E, E^\sigma)$, $\sigma$ acts on $(P, f(P))$ by $(P, f(P))^\sigma = ((f(P))^\sigma, P^\sigma) = (f^\sigma(P^\sigma), P^\sigma)$.
Note here that \( f^{\sigma} \) is an isogeny in \( \text{Hom}_K(E^\sigma, E) \) which has the same degree as \( f \). Note also that a graph cycle class of an isogeny \( f \)

\[ [G_f]_{\text{alg}} = \left\{ \{(P, f(P)) \mid P \in E\} \right\}_{\text{alg}} \]

is fixed by \( \langle \sigma \rangle \) if and only if the following condition holds:

\[ \left\{ (f^{\sigma}(P^\sigma), P^\sigma) \mid P^\sigma \in E^\sigma \right\}_{\text{alg}} \cong \left\{ (P, f(P)) \mid P \in E \right\}, \]

that is, \( G_{f^{\sigma}} \) coincides with \( G_f \) modulo algebraic equivalence. Using the fact that the graph of \( f \) and the graph of \( \hat{f} \) are always algebraically equivalent, we can replace the condition above by

\[ \left\{ (f^{\sigma}(P^\sigma), P^\sigma) \mid P^\sigma \in E^\sigma \right\}_{\text{alg}} \cong \left\{ (\hat{f}(P^\sigma), P^\sigma) \mid P^\sigma \in E^\sigma \right\}, \]

i.e.

\[ f^\sigma = \hat{f}. \]

Now we are treating non-CM elliptic curve \( E \). Hence actually \( f^\sigma = \hat{f} \) holds up to the sign because \( f^\sigma \circ f \in \mathbb{Z} \cong \text{End}_K(E) \). But in the local case, the analogous phenomenon may not occur. In fact, from a point of view to apply the method of Midhahi and Flach the ordinary case is done by the same argument before but the supersingular case cause some difficulty if the galois conjugate isogenies and the dual isogenies coincides completly.

Let’s see this problem more precisely. Let \( p \) be supersingular prime and take the \( \mathbb{Z} \)-basis \( \phi_p, \phi_p \circ \alpha_p, \phi_p \circ \beta_p, \phi_p \circ \alpha_p \circ \beta_p \) for the homomorphism group \( \text{Hom}_{\mathbb{F}_p}(E_p, E_p^\sigma) \) (isomorphic to an order of a definite quaternion algebra). Here \( \phi_p \) is the reduction isogeny of \( \phi, \langle \phi \rangle = \text{Hom}_K(E, E^\sigma) \cong \mathbb{Z}, \alpha_p, \beta_p, \alpha_p \circ \beta_p \) is “the quaternion” endomorphism in \( \text{End}_{\mathbb{F}_p, e}(E_p) \).

We can directly deduce that \( \phi_p^\sigma = \widehat{\phi_p} \) up to sign. To decide whether or not the equality holds between the galois conjugate and the dual for other three isogenies is difficult, i.e. whether \( f_p^\sigma = \hat{f}_p \) holds or not up to sign for \( f_p = \phi_p \circ \alpha_p, \phi_p \circ \beta_p, \phi_p \circ \alpha_p \circ \beta_p \) is conjectured to be a characteristic phenomenon of the \( \mathbb{Q} \)-curve \( E \).

For this reason, we need the following assumption:

\[
\begin{align*}
(\phi_p \circ \alpha_p)^\sigma & = \widehat{\phi_p \circ \beta_p}, \\
(\phi_p \circ \beta_p)^\sigma & = \widehat{\phi_p \circ \alpha_p \circ \beta_p}, \quad \text{for every inert and supersingular prime } p \\
(\phi_p \circ \alpha_p \circ \beta_p)^\sigma & = \widehat{\phi_p \circ \alpha_p},
\end{align*}
\]

up to the sign. By the assumption, the graph cycle class \( [G_{\phi_p}]_{\text{alg}} \) is in the galois fixed part \( \text{DC}_{\mathbb{F}_p}(E_p, E_p^\sigma)_{\text{alg}}(\sigma_p) \), but the graph cycle classes \([G_{\phi_p \circ \alpha_p}]_{\text{alg}}, [G_{\phi_p \circ \beta_p}]_{\text{alg}} \) and \([G_{\phi_p \circ \alpha_p \circ \beta_p}]_{\text{alg}} \) are not in the galois fixed part. This means we can finish the claim:

\[ \text{Coker}(\partial_p) := \text{Coker}(\partial_p : H^1(A, K_2) \to \text{Pic}(A_p)) \]

is torsion for any \( p \) which is inert in \( K \).
and our aim that $\text{Coker}(\partial)$ is torsion is completed. That is, we get the main theorem A.

**Theorem A.** Let $E$ be a modular elliptic $\mathbb{Q}$-curve over a quadratic field $K$. Assume that $E$ is isogenous over $K$ to its galois conjugate $E^\sigma$ and that $E$ satisfies the condition $(\ast)$. Let $A := \text{Res}_{K/Q}(E)$ be its Weil restriction. Then for any fixed integer $n \neq 1$ the $n$-torsion subgroup $n\text{CH}^2(A)$ of the second Chow group $\text{CH}^2(A)$ is finite.

**References**


